8. E. J. Hinch and L. G. Leal, "The effect of Brownian motion of the rheological properties of a suspension of non-spherical particles," J. Fluid Mech., 52, No. 4 (1972).
9. L. G. Leal and E. J. Hinch, "The effect of weak Brownian rotations on particles in shear flow," J. Fluid Mech., 46, No. 4 (1971).
10. V. N. Pokrovskii, Statistical Mechanics of Dilute Suspensions [in Russian], Nauka, Moscow (1978).
11. W. Wasow, Asymptotic Expansions of Solutions of Ordinary Differential Equations [Russian translation], Mir, Moscow (1968).
12. A. J. McConnell, Introduction to Tensor Analysis with Applications of Geometry, Mechanics and Physics [Russian translation], Fizmatgiz, Moscow (1963).

CONVECTIVE INSTABILITY IN A MEDIUM WITH SPIRAL TURBULENCE

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In the papers of Moiseev, Sagdeev, Tur, et al. (see [1] and the literature cited there), the generation of large-scale convective structures on a background of spiral turbulence was considered and the relevant equations were obtained and analyzed. It was assumed that the turbulence is homogeneous and isotropic but does not possess reflection invariance. In this model random perturbations are amplified, which can lead to generation of large-scale vortices. This situation was studied in [1] for the example of a plane-parallel layer of incompressible liquid heated from below. Simplified boundary conditions were assumed in order to obtain an analytic solution. It was shown that as the spirality increases the minimum critical Rayleigh number decreases and the horizontal dimensions of the convection cells increase. At a critical value of the spirality, the structure of the convective flow change completely and a vortex is formed whose dimensions are determined by the external conditions of the problem such as inhomogeneities in the horizontal direction.

In the present paper the equations of [1] are used to analyze the convective instability in an infinite horizontal layer and in a disk heated from below in the linear theory.

The equations describing convection for large-scale disturbances in the presence of spiral turbulence have the form [1]

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla p}{\rho_0} + \nu\Delta u + \beta g\theta\mathbf{e} + \beta g A\lambda\mathbf{f},\ \frac{\partial\theta}{\partial t} = A(\mathbf{eu}) + \chi\Delta\theta,\ \operatorname{div}\mathbf{u} = 0,$$
$$\mathbf{f} = \mathbf{e}(\mathbf{e}\operatorname{rot}\mathbf{u}) - (\mathbf{e}\nabla)[\mathbf{eu}],\ \mathbf{e} = (0, 0, 1).$$

Here ν and χ are the turbulent viscosity and thermal conductivity. Because these quantities are nearly equal to one another [1-3], we assume $\nu = \chi$. The coefficient λ is associated with the spirality of the turbulence. The rest of the notation is standard [2].

We transform to dimensionless variables using as scales of measurement [2]: $x_0 = H$ (height of the liquid layer) for length, $t_0 = H^2/\nu$ for time, $u_0 = \nu/H$ for velocity, $p_0 = \rho_0\nu^2/H^2$ for pressure, and $T_0 = AH$ for temperature. Then

$$\partial\mathbf{u}/\partial t = -\nabla p + \Delta\mathbf{u} + \mathrm{Ra}\,\theta\mathbf{e} + \mathrm{Ra}\,S\mathbf{f},\ \partial\theta/\partial t = (\mathbf{eu}) + \Delta\theta,$$
$$\operatorname{div}\mathbf{u} = 0,\ \mathbf{f} = \mathbf{e}(\mathbf{e}\operatorname{rot}\mathbf{u}) - (\mathbf{e}\nabla)[\mathbf{eu}],\ \mathbf{e} = (0, 0, 1), \tag{1}$$

where Ra is the Rayleigh number and S is a coefficient connected with the spirality of the turbulence ($\mathrm{Ra} = \beta gAH^4/\nu^2$, $S = \lambda\nu/H^3$).

We consider an infinite horizontal liquid layer included between two planes $z = 0$ and $z = 1$. Then $\mathbf{u} = (u, v, w)$ and θ are given by

$$u = u'(z)\sin kx\exp(\gamma t),\ v = v'(z)\sin kx\exp(\gamma t),$$
$$w = w'(z)\cos kx\exp(\gamma t),\ \theta = \theta'(z)\cos kx\exp(\gamma t).$$

In the case of a cylindrical layer (disk) $\cos kx$ is replaced by the Bessel function of order zero $J_0(kr)$ and $\sin kx$ is replaced by the Bessel function of order one $J_1(kr)$. The results given below do not change in this case. Putting these substitutions into (1) we find

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(we omit the primes on u', v', w', θ'):

$$(\gamma + k^2) u - \frac{d^2u}{dz^2} = kp + \mathrm{Ra}\, S\frac{dv}{dz}, \ (\gamma + k^2) v - \frac{d^2v}{dz^2} = -\mathrm{Ra}\, S\frac{du}{dz},$$
$$(\gamma + k^2) w - \frac{d^2w}{dz^2} = -\frac{dp}{dz} + \mathrm{Ra}\theta + \mathrm{Ra}\, Skv, \ (\gamma + k^2)\theta - \frac{d^2\theta}{dz^2} = w, \ \frac{dw}{dz} + ku = 0. \quad (2)$$

For ordinary convection in a horizontal layer with free boundaries the eigenfunctions are proportional to $\sin n\pi z$ and the vertical component of the velocity vanishes at z = 0 and z = 1. In the case of convection with spiral turbulence these eigenfunctions correspond to periodic boundary conditions in the z direction. Substituting in (2) eigenfunctions $w \sim \sin\pi z$, $u \sim \cos\pi z$, and so on which correspond to the lowest harmonic, we obtain the following expression for the increment for wave number k [1]

$$\gamma = \left(\frac{\mathrm{Ra}^2 S^2\pi^2(\pi^2 - k^2) + \mathrm{Ra}k^2}{\pi^2 + k^2}\right)^{1/2} - (\pi^2 + k^2).$$

Putting $\gamma = 0$, we find the critical Rayleigh number Ra_0 as a function of k:

$$\mathrm{Ra}_0 = \frac{(k^4 + 4\pi^2(\pi^2 + k^2)^3(\pi^2 - k^2)S^2)^{1/2} - k^2}{2\pi^2(\pi^2 - k^2)S^2}.$$

The dependence $\mathrm{Ra}_0(k, S)$ was analyzed in [1].

The spirality coefficient can be defined as $s = \mathrm{Ra}\,S$. This form is more convenient in studying the properties of the neutral stability curves. In this case

$$\gamma = \left(\frac{s^2\pi^2(\pi^2 - k^2) + \mathrm{Ra}\, k^2}{\pi^2 + k^2}\right)^{1/2} - (\pi^2 + k^2), \ \mathrm{Ra}_0 = \frac{(\pi^2 + k^2)^3 - s^2\pi^2(\pi^2 - k^2)}{k^2}. \quad (3)$$

The neutral stability curves $\mathrm{Ra}_0(k, s)$ are shown in Fig. 1. Curves 1-5 correspond to $s = 0, 0.8\pi, \pi, 1.3\pi, 2\pi$, respectively. As s increases from 0 to π minima of the neutral stability curves shift toward longer wavelengths and the minimum Rayleigh number decreases. As s approaches π the minimum Rayleigh approaches $4\pi^4 \approx 389$. The value of s equal to $s_* = \pi$ and Ra equal to $\mathrm{Ra}_* = 4\pi^4$ define regions with distinct properties: for $s < s_*$ there always exists a region of stable long-wave harmonics and when $s = s_*$ and $\mathrm{Ra} > \mathrm{Ra}_*$ such a region does not exist, i.e., even infinite-wavelength perturbations become unstable. When $s > s_*$ the neutral stability curves do not have minimum points and waves with k = 0 will be unstable for any Ra. In terms of the initial definition of the spirality this means that for $S\,\mathrm{Ra} > S_*\mathrm{Ra}_*$ ($S_* = s_*/\mathrm{Ra}_*$) perturbations with k = 0 become unstable, i.e.,

$$\mathrm{Ra}_0(k = 0, S) = \mathrm{Ra}_* S_*/S. \quad (4)$$

In the case of different boundary conditions the values of Ra_* and S_* change, but the nature of the neutral stability curves and the dependence on k and s remains the same. Therefore (4) is universal for any boundary conditions.

Since we are mainly interested in the generation of long-wave disturbances, we consider the conditions for which the maximum value of the increment γ will lie in the long-wave region ($k \ll \pi$), i.e., we study the behavior of $\gamma(\mathrm{Ra}, s)$ at small k. It is easy to show that when $|s - s_*| \ll s_*$ and $k^2 \ll \pi^2$ the increment and $\mathrm{Ra}_0(k, s)$ become

$$\gamma = \alpha(s - s_*) + \eta\frac{\mathrm{Ra} - \mathrm{Ra}_*}{\mathrm{Ra}_*}k^2, \ \mathrm{Ra}_0 = \mathrm{Ra}_*\left(1 - \frac{\alpha}{\eta}\frac{s - s_*}{k^2}\right) \ (\alpha = \pi, \ \eta = 2). \quad (5)$$

In terms of the original definition of the spirality we obtain for $\mathrm{Ra}_0(k, S)$

$$\mathrm{Ra}_0 = \mathrm{Ra}_*\frac{S_*}{S}\left(1 + \frac{\eta}{\alpha\mathrm{Ra}_*}\frac{S - S_*}{S_*^2}k^2\right).$$

We consider the case of free boundary conditions at z = 0 and z = 1. In the theory of ordinary convection free boundaries are defined to be boundaries at which tangential stresses vanish, i.e., the conditions $\partial u/\partial z = \partial v/\partial z = 0$ are satisfied. The equations describing convection in the presence of spiral turbulence differ from the Bousinessque equations used to construct the theory of ordinary convection, therefore the conditions for free boundaries must be reformulated. We write (1) in the form

$$\partial u_i/\partial t = -\partial\sigma_{ik}/\partial x_k, \ \sigma_{ik} = p\delta_{ik} - (\partial u_i/\partial x_k + \partial u_k/\partial x_i) + \mathrm{Ra}\, S(e_i\varepsilon_{kmn} + e_k\varepsilon_{imn})u_m e_n.$$

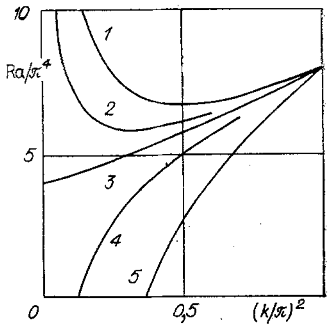

Fig. 1

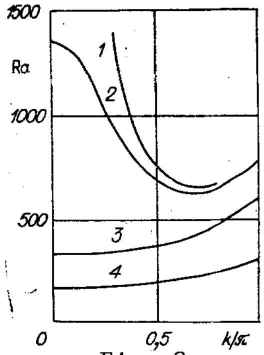

Fig. 2

We define a free boundary as one at which the tangential stresses vanish, i.e., $\sigma_{xz} = \sigma_{yz} = 0$. Because $\sigma_{xz} = \partial u/\partial z + \mathrm{Ra}\, Sv$, $\sigma_{yz} = \partial v/\partial z - \mathrm{Ra}\, Su$, the free boundary conditions in the presence of spirality take the form $\partial u/\partial z = -\mathrm{Ra}\, Sv$, $\partial v/\partial z = \mathrm{Ra}\, Su$.

We assume as before that θ is equal to zero on the boundary. The problem of the neutral stability curves has not been solved completely in this case, but it is possible to find Ra_*, S_*, α, η. To do this it is convenient to adopt the following change of notation: $u = u'$, $v = sv'$, $w = kw'$, $\theta = k\theta'$. We introduce the vorticity $f = \partial u'/\partial z + k^2 w'$. Then the system of equations (2) can be rewritten as (the primes on u', v', w', θ' are omitted)

$$\begin{gathered}\gamma f = \partial^2 f - k^2 f + s^2(\partial^2 v + k^2 v) + \mathrm{Ra}\, k^2\theta,\\ \gamma v = \partial^2 v - k^2 v + \partial^2 w,\\ \gamma\theta = \partial^2\theta - k^2\theta + w,\ \partial^2 w - k^2 w = -f,\ \partial \equiv \partial/\partial z.\end{gathered} \tag{6}$$

The boundary conditions at $z = 0$ and $z = 1$ are

$$\theta = w = f + s^2 v = \partial v + \partial w = 0. \tag{7}$$

It will be convenient to define a function ψ such that $\psi : \partial\psi = v$. Then with the help of (7) we can rewrite the second equation of (6) as $\gamma\psi = \partial^2\psi - k^2\psi + \partial w$. The boundary conditions will be $\theta = f + s^2 v = \psi = 0$.

We calculate s_* and α, putting $k^2 = 0$. Let $s = s_* + \mu$, where $|\mu| \ll s_*$. Then the increment γ is of order μ. We assume a solution of (6) in the form of an expansion in this small parameter. The zeroth approximation has the form

$$\partial^2 f_0 + s_*^2 \partial^2 v_0 = 0,\ \ \partial^2\psi_0 + \partial w_0 = 0,\ \ \partial^2 w_0 = -f_0,\ \ v_0 = \partial\psi_0.$$

The boundary conditions in the zeroth approximation are $w_0 = \psi_0 = f_0 + s_*^2 v_0$. Then it is easy to show that

$$\begin{gathered}v_0 = \cos s_*(z - 1/2),\ \ f_0 = -s_*^2 v_0,\\ w_0 = -(v_0 + 1),\ \ \psi_0 = s_*^{-1}\sin s_*(z - 1/2),\ \ s_* = 2\pi.\end{gathered} \tag{8}$$

The first approximation is described by the equations

$$\begin{gathered}\partial^2 f_1 + s_*^2\partial^2 v_1 = \gamma f_0 - 2s_*\mu\partial^2 v_0,\\ \partial^2\psi_1 + \partial w_1 = \gamma\psi_0,\ f_1 = -\partial^2 w_1,\ v_1 = \partial\psi_1.\end{gathered}$$

The boundary conditions at $z = 0$ and $z = 1$ take the form

$$w_1 = \psi_1 = f_1 + s_*^2 v_1 + 2s_*\mu v_0 = 0,$$

hence

$$\begin{gathered}f_1 + s_*^2 v_1 = \gamma F - 2s_*\mu v_0,\\ v_1 + w_1 = \gamma\int\psi_0 dz + C,\ f_1 = -\partial^2 w_1.\end{gathered} \tag{9}$$

Here F is the second antiderivative of f_0 which vanishes at $z = 0$ and $z = 1$. It is easily seen that $F = v_0 + 1$ [see (8)]. We let the integral $\int\psi_0 dz$ be given by the function $(-v_0/s_*^2)$ and C is the constant of integration. We then obtain from (9)

$$\partial^2 w_1 = -s_*^2 w_1 + 2(s_*\mu - \gamma)v_0 - \gamma + s_*^2 C.$$

The solvability of this equation will give the increment γ. We multiply this equation by w_0 and integrate from 0 to 1. Evaluating the resulting integral on the left-hand side by parts

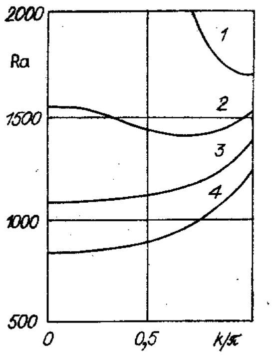

Fig. 3

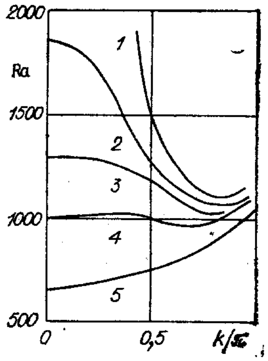

Fig. 4

using the boundary conditions and (8), we obtain the expression

$$-s_*^2\int_0^1 w_1 dz = 2\gamma - s_*\mu - s_*^2 C.$$

Because $\int_0^1 v_1 dz = \int_0^1 \partial\psi_1 dz = 0,$ integrating the second equations of (9), we obtain

$$\int_0^1 w_1 dz = C.$$

Hence we finally obtain $\gamma = s_*/2\mu \equiv \pi(s - s_*)$ and therefore $\alpha = \pi$.

We calculate η and Ra_*, putting $s = s_*$, $v = v_0 + k^2 v_1$, $w = w_0 + k^2 w_1$, $\theta = \theta_0 + k^2\theta_1$, $\gamma = \xi k^2$, and so on. We have (8) for v_0, w_0, f_0, and ψ_0 and for θ_0 we have the equation $\partial^2\theta_0 = -w_0$ with the boundary conditions $\theta_0(0) = \theta_0(1) = 0$, which is easily solved. The first approximation in k^2 is

$$\partial^2 f_1 + s_*^2\partial^2 v_1 = (1+\xi) f_0 - s_*^2 v_0 - \mathrm{Ra}\,\theta_0,$$
$$\partial^2 v_1 + \partial^2 w_1 = (1+\xi) v_0,\ \partial^2 w_1 + f_1 = w_0.$$

The further steps are analogous to those for the calculation of α. The solvability condition for the above equations gives

$$\xi = \frac{5}{2}\left[\mathrm{Ra}\frac{s_*^4 + 20 s_*^2 + 420}{600 s_*^4} - 1\right].$$

Because $\gamma = \xi k^2 = \eta\frac{\mathrm{Ra} - \mathrm{Ra}_*}{\mathrm{Ra}_*}k^2$, we have

$$\eta = 5/2,\ \mathrm{Ra}_* = 600 s_*^4/(s_*^4 + 20 s_*^2 + 420) \approx 337.8,\ S_* = s_*/\mathrm{Ra}_* \approx 1.86\cdot 10^{-2}.$$

In the general case the neutral stability curves can be found numerically. In Fig. 2 curves 1-4 correspond to $S = 0$, $0.25S_*$, S_*, $2S_*$. When the spirality increases from 0 to S_* the minima of the neutral stability curves shift toward longer wavelengths and the value of the minimum decreases. When $S \geqslant S_*$ the minimum is reached at the point $k = 0$ and is equal to Ra_*S_*/S [compare (4)].

We consider the case when the velocities vanish at both boundaries: $u = v = w = 0$. Calculations similar to above then give $s_* = 2\pi$, $\alpha = \pi$, $\eta = 2$, $\mathrm{Ra}_* = 384\pi^4/(2\pi^2 + 15) \approx 1077.96$, $S_* \approx 5.83\cdot 10^{-3}$. The neutral stability curves for these boundary conditions were calculated numerically and are shown in Fig. 3, where curves 1-4 correspond to $S = 0$, $S = 0.7S_*$, S_*, $1.3S_*$.

In the case where the velocities vanish on the lower boundary ($z = 0$), while the upper boundary ($z = 1$) is free, we have

$$s_* = \mathrm{tg}\, s_* \approx 4.4934,\ \alpha = \frac{3 s_*}{4 + C} \approx 1.1756,$$

$$\mathrm{Ra}_* = (8 + C)\frac{90 s_*^4}{315 + 6 s_*^2} \approx 1301.12,\ \eta = \frac{8 + C}{4 + C} \approx 1.3488,$$

$$S_* = s_*/\mathrm{Ra}_* \approx 3.4535\cdot 10^{-3},$$

where $C = 3\left(\frac{s_*}{6} + A\right)A,\ A = \frac{1}{s_*} - \left(1 + \frac{1}{s_*^2}\right)\sin s_*$.

The neutral stability curves are shown in Fig. 4, where curves 1-5 correspond to S = 0, $0.7S_*$, S_*, $1.3S_*$, $2S_*$. We note that $Ra_* \approx 1301$ is greater than the minimum Rayleigh number at S = 0, which is approximately equal to 1100. Therefore at $S = S_*$ the point k = 0 is a local minimum and disturbances with k = 0 become most unstable for large S: $S \gtrsim 1.4S_*$.

From the cases analyzed here, we conclude that when the spirality increases from 0 to a certain value S_* the minimum of the neutral stability curves $Ra_0(k, S)$ shifts in the direction of smaller wave numbers k, and hence the horizontal dimensions of the convection cells increase. When $S \geqslant S_*$ the minimum is reached at k = 0 and the horizontal dimensions of the cells are limited by the external conditions (for example, inhomogeneities in the horizontal direction).

LITERATURE CITED

1. S. S. Moiseev, P. B. Putkevich, A. V. Tur, and V. V. Yanovskiĭ, "Vortex dynamo in a convective medium with spiral turbulence," Zh. Éksp. Teor. Fiz., 94, No. 2 (1988).
2. G. Z. Gershuni and E. M. Zhukhovitskiĭ, Convective Instability of an Incompressible Fluid [in Russian], Nauka, Moscow (1972).
3. A. S. Monin and A. M. Yaglom, Statistical Fluid Mechanics [in Russian], Nauka, Moscow (1965), Part 1.

EFFECT OF INTERPHASE MASS TRANSFER ON THE TURBULENCE ENERGY OF A FLOW OF A GAS SUSPENSION

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Semiempirical turbulence models based on equations describing the transfer of fluctuation energy are widely used to calculate flows of gas suspensions (see [1-3] and the accompanying bibliographies). Here, we attempt to use these models to describe the flows of gas suspensions with phase transformations (such as in [3], where allowance was made for the heterogeneous combustion of dispersed particles). We will analyze the direct effect of interphase mass transfer on the turbulence energy of the dispersion medium.

1. Equation of Turbulence Energy Transfer. In the presence of phase transformations, the equations of conservation of mass and momentum for the carrier phase (dispersion medium) are written as follows [4]

$$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{V}) = J; \tag{1.1}$$

$$\rho d\mathbf{V}/dt = -\nabla p + \nabla\tau - \mathbf{F} + J(V_p - V), \tag{1.2}$$

where ρ and $\mathbf{V}$ are the distributed density and velocity of the dispersion medium, the subscript p denotes the disperse phase, J is the intensity of the interphase mass transfer, p is pressure, τ is the shear stress, $\mathbf{F}$ is the interfacial force.

Using (1.1) and (1.2) and a well-known procedure (see [3], for example), we can obtain an equation for the fluctuation energy of the dispersion medium k. If we ignore fluctuations of the density of the gas ρ', this equation has the form

$$\rho\mathbf{V}\nabla k = \nabla[\mu\nabla k - \rho\langle\mathbf{V}'(\tfrac{1}{2}\mathbf{V}'^2 + p'/\rho)\rangle] - \rho\langle\mathbf{V}'\mathbf{V}'\rangle\nabla\mathbf{V} + \tfrac{1}{3}\mu\langle\mathbf{V}'\nabla(\mathrm{div}\ \mathbf{V}')\rangle + \langle p'\nabla\mathbf{V}'\rangle - \rho(\varepsilon + \varepsilon_p + \varepsilon_J) \tag{1.3}$$

Here and below, the primes denote the fluctuation component, while the remaining terms are averaged over time; $\rho\varepsilon = \mu\sum_{ij}\langle(\partial V_i'/\partial x_j)^2\rangle$ is the rate of viscous dissipation of the turbulence energy, μ is the coefficient of dynamic viscosity of the gas, $\rho\varepsilon_p = \sum_i\langle F_i'V_i'\rangle$ is a dissipative term due to the dynamic interaction of the phases and fluctuation motion.

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